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A fast unbinned test on event clustering in Poisson processes

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Abstract. An unbinned statistical test on cluster-like deviations from Poisson processes for point process data is introduced, presented in the context of time variability analysis of astrophysical sources in count rate experiments. The measure of deviation of the actually obtained temporal event distribution from that of a Poisson process is derived from the distribution of time differences between two consecutive events in a natural way. The differential character of the measure suggests this test in particular for the search of irregular burst-like structures in experimental data. The construction allows the application of the test even for very low event numbers. Furthermore, the test can easily be applied in the case of varying acceptance of the detector as well. The simple and direct use of background events simultaneously acquired under the same conditions to account for acceptance variations is possible, allowing for easy application especially in earth-bound γ-ray experiments. Central features are the fast and simple calculation of its measure, and the existence of an analytical approximation that describes the general test statistics to a high degree of precision.

Key words: methods: data analysis – methods: statistical

1. Introduction

In the search for high energy astrophysical γ-ray sources, be it with earth-bound experiments or with satellite borne detectors, the test on an integral (DC) excess of event numbers from the direction of a source, compared to an adequately derived background expectation, is certainly the major statistical tool. (For a quite general consideration of this topic, see Li & Ma 1983). However, an integral excess of events is not the only characteristic one might be interested in. Many γ-ray sources are known to exhibit highly variable fluxes.

In this paper a newly developed test on variability for photon counting experiments is presented. In contrast to time series analyses, in which sampled continuous functions are examined and which are applied e. g. in optical astronomy, this test deals with discrete registration times of single events, i. e. *point process data*. This situation is typically encountered in γ-ray astronomy.

It is dedicated to two basic, different, but closely related questions that require statistical methods:

1. Are there significant signals of variability from a certain celestial position even if there is only a moderate or hardly significant DC excess from that direction, if the observation is limited by background fluctuations?
2. How well is the sequence of arrival times of pure (or almost pure) events from a well-known source described by a constant flux?

Considering the case of known or assumed periodicities of activity of the objects in question, several statistical tests are established to search for fixed frequencies in point process data (see e. g. Mardia 1972, Lewis 1994).

For the remaining case of irregular temporal activity schemes, the situation concerning established tests is less satisfying. In many publications, more or less spontaneously invented measures of variability are used, based mostly on a procedure of binning the event data in equally sized time slices and subsequent application of a χ^2 test.

Based on this idea, some rather elaborated tests have been worked out, binning the data into a lot of different ways under variation of bin phases and widths (e. g. Collura et al. 1987, Biller et al. 1994). The measure of variability is then obtained by a certain combination of all single χ^2 test results. Such tests typically suffer from two deficiencies:

1. The chance probability for a given probe can in general only be assessed by M$^{\rm te}$ Carlo means, since the single χ^2 test results are not independent
2. The computation of the measure itself is quite expensive expressed in terms of computing time and storage needs since the data has to be divided into a lot of different binning schemes

Other drawbacks in certain situations are the restriction of those tests to sufficiently high event statistics, and the fact that the rigorous employment of simultaneously acquired background events to account for a non-uniform temporal

Send offprint requests to: J. Prahl, prahl@mail.desy.de acceptance of a detector is not obvious. In addition, the procedures are by far too complicated to apply them to a huge number of different potential source directions, e. g. using them for a kind of all-sky search for variable sources.

It should not be concealed that there exist alternative variability analysis procedures for point process data (see e. g. Scargle 1998 or Gregory & Loredo 1996 for an application of Bayesian methods). These are surely appropriate to attack "higher order questions" such as typical time scales of variability or change points of fluxes, but the application is at least as complicated as the use of binned tests.

These difficulties may well bring down the willingness of experimenters to apply those tests. The unbinned *exp-test* that will be introduced in the following is, in contrast to the tests mentioned above, easy to apply, using a straightforward, natural measure for variability. The result of it will therefore not be as specific as for most of the above tests, but will supply a measure for variability in general.

It should be emphasized that one established test of this kind is already existing: the nonparametric Kolmogorov test, applied to the cumulated distribution function (cdf) of the registration times of the events in question, comparing them to the expected equal distribution. In opposition to it, the exp-test will have a more differential character, thus complementing the Kolmogorov test in a sense that will be specified later.

Although described here in the context of time variability analysis, the application of the exp-test is not restricted to the time domain, but may be used in the analysis of e. g. spatial or frequency data as well. It might also be found useful in other disciplines of science where the analysis of empirical data plays a role, like biology, economics and engineering.

In the following section, the exp-test is developed for the ideal situation of a uniform or exactly known temporal acceptance function of the experiment with respect to the events from a potential source. Section 3 deals with the generalization needed to apply the exp-test with simultaneously registered background events, while in section 4 the sensitivity of the test is characterized. In section 5, a brief comparison with the Kolmogorov test is carried out, and in the last section a summary is given.

2. Developing the exp-test

Supposed a uniform temporal acceptance of the used detector is given, the expected temporal sequence of registered events in absence of a variable source is governed by Poisson statistics. This applies to both background events and events from a steady source, thereby defining the zero hypothesis for *any* test on variability in this context.

One starts with the Poisson distribution

$$P_\lambda(n) = e^{-\lambda} \cdot \frac{\lambda^n}{n!} \tag{1}$$

where λ denotes the expectation value.

A monotone sequence (t_i) of events in time t is called a *Poisson process*, if there exists a constant C, such that for all $\Delta t > 0$, dividing the time in equally sized intervals of the length Δt, the numbers of events per time interval are Poisson distributed with $\lambda = \Delta t / C$ and mutually independent (see e. g. Chatfield 1994). From this immediately follows the probability density function (pdf) of the time intervals Δt between two consecutive events as the derivative of $P_\lambda(0)$ with respect to Δt

$$\begin{aligned} f_C(\Delta t) &= -\frac{P_{(\Delta t + d\Delta t)/C}(0) - P_{(\Delta t)/C}(0)}{d\Delta t} \\ &= -\frac{d \exp(-\Delta t / C)}{d\Delta t} = \frac{1}{C} \cdot \exp\left(-\frac{\Delta t}{C}\right) \end{aligned} \tag{2}$$

i. e. an exponential distribution in Δt with the mean value C.

Let there now be a randomly chosen time interval, containing the monotonically increasing sequence of event times $(T_i)_{i=1\ldots N+1}$. Let the resulting distribution

$$\{\Delta T_i\}_{i=1\ldots N} := \{(T_{i+1} - T_i)\}_{i=1\ldots N} \tag{3}$$

have a mean value $\overline{\Delta T} =: C^*$. If one sets $C = C^*$ in Eq. 2 (i. e. setting C to the actually obtained mean value), it follows for the ΔT_i to stem from a distribution $f_{C^*}(\Delta t)$, under the constraint of the conserved mean value. This is generally valid, even for *a priori* unlikely values of C^*, because the T_i are completely uncorrelated. This ensures that any analysis based only on C^* is independent of a possibly present DC excess or deficit.

Now the distribution of the ΔT_i can be compared with $f_{C^*}(\Delta t)$ using a distribution test. The classical tests (e. g. Kolmogorov test or Smirnov–Cramér–von-Mises test) are neither well suited to measure exactly the effect that is searched for, nor is it possible to take the constraint of the conserved mean value into account. For this reason a natural measure will be introduced here that is especially sensitive to excesses of the ΔT_i far from the mean value.

Defining the pdf $\mathrm{F}(\Delta t)$ of the random probe

$$\mathrm{F}(\Delta t) := \frac{1}{N} \sum_{i=1}^{N} \delta(\Delta t - \Delta T_i) \tag{4}$$

the equalities of the 0. and 1. momenta of f_{C^*} and F serve as a starting point:

$$\int_0^\infty f_{C^*}(\Delta t)\, d\Delta t = \int_0^\infty \mathrm{F}(\Delta t)\, d\Delta t = 1 \tag{5}$$

$$\int_0^\infty \Delta t \cdot f_{C^*}(\Delta t)\, d\Delta t = \int_0^\infty \Delta t \cdot \mathrm{F}(\Delta t)\, d\Delta t = C^* \tag{6}$$

The variance of the distribution $\mathrm{F}(\Delta t)$ could be used as a measure for event clusters (and corresponding dilutions), but this quantity is too sensitive on very large ΔT_i due to the extreme assymmetry of the exponential distribution. (Actually the value of the variance is dominated by the pdf between C^* and ∞, although only 37 % of the ΔT_i are contained in that interval.) In order to be not too sensitive to outliers, another way will be followed. From Eqs. 5, 6 follows

$$\int_0^\infty \underbrace{\left(1-\frac{\Delta t}{C^*}\right)}_{:=h(\Delta t)} \cdot f_{C^*}(\Delta t)\, d\Delta t$$
$$= \int_0^\infty \left(1-\frac{\Delta t}{C^*}\right) \cdot \mathrm{F}(\Delta t)\, d\Delta t = 0 \tag{7}$$

Defining now

$$\mathrm{H}(x) := \int_0^x h(\Delta t) \cdot f_{C^*}(\Delta t)\, d\Delta t \tag{8}$$

it can be concluded

- $\mathrm{H}(0) = \mathrm{H}(\infty) = 0$
- $\mathrm{H}(C^*) = 1/e$ ist the global maximum of H, as f_{C^*} is positive everywhere, and $h(\Delta t)$ is monotonically falling, being zero at C^*

Replacing in Eq. 8 f_{C^*} by F, the first property holds, and the global maximum is again found at C^*, but varying around a mean value near $1/e$. (The transition from C, i. e. the expected value, to C^*, the actual mean value of the random probe, slightly reduces this value.[1] This fact will be investigated later.) Hence one has

$$\mathrm{M(F)} := \int_0^{C^*} \left(1-\frac{\Delta t}{C^*}\right) \cdot \mathrm{F}(\Delta t)\, d\Delta t \tag{9}$$

(with $C^* = \int \Delta t\, \mathrm{F}(\Delta t)\, d\Delta t$)

being a functional on the set of all pdf defined on $[0, \infty[$, with the property $\mathrm{M}(f_C) = 1/e$.

Provided C^* is already computed, M can be calculated from the pdf of the ΔT_i between 0 and C^* alone. It will be higher than expected if untypical excesses of small ΔT_i are present (i. e. in the case of burst-like phenomena), and lower than expected for untypical excesses of ΔT_i near C^* (i. e. for untypically regular temporal structures compared to the average Poisson process).

Fixing the extreme cases

1. the heartbeat function
$\heartsuit(\Delta t) := \delta(\Delta t - C^*)$
belonging to the case of all ΔT_i being equal, leading to $\mathrm{M}(\heartsuit) = 0$, and
2. the needle peak function
$\spadesuit(\Delta t) := \lim_{\varepsilon\to 0}(1-\varepsilon)\cdot\delta(\Delta t) + \varepsilon\cdot\delta(\Delta t - C^*/\varepsilon))$
characterizing the case of almost all $\Delta T_i = 0$, and the rest (e. g. one of them) being huge, resulting in the mean value of C^*, which leads to $\mathrm{M}(\spadesuit) \to 1$.

it can be noted for any pdf on $[0, \infty[$ with a mean value of C^*:

$$\mathrm{M(F)} \in [0, 1[\tag{10}$$

From the above it can be concluded that M defines a well-behaved and natural measure of variability, therefore M is chosen to be the measure of the exp-test. It should be stressed that only the spectrum of time intervals *between* two consecutive events is entering, thus giving this test a rather differential character.

It remains the derivation of the actual distribution function for the M(F) for given N (in the following denoted M_N distribution) under the zero hypothesis. In any case, this pdf is independent on the particular value of C^*, since this constant only scales the time, whereas M is a dimensionless quantity. Analytically exact solutions are surely possible but not attempted here, because a considerable expenditure has to be expected. Instead, a semi-analytical approach is followed, using the fact that all spreads occuring here are resulting from spreads in the polynomial distribution, all of them scaling with $\sqrt{N}$, thus leading for dimensionless variables such as M to asymptotic standard deviations $\sigma \propto 1/\sqrt{N}$. For a similar reason, the difference of the mean value μ and $1/e$ will be, again at least asymtotically, proportional to $1/N$. Furthermore it can be expected, according to the central limit theorem

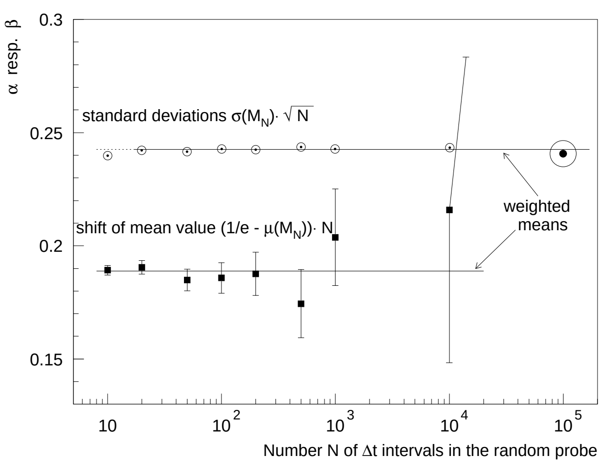

Fig. 1. The standard deviations $\sigma(\mathrm{M}_N)\cdot\sqrt{N}$ and the shift of the mean value $(1/e - \mu(\mathrm{M}_N))\cdot N$ of the M_N distribution from M^{te} Carlo calculations. Shown are also the weighted mean values (straight lines). The errors for the spread values (circles) are indicated by the size of the central spot. Sample sizes are $1.3\cdot 10^5$ for each tested value of N (except for $N = 10^5$ with a sample size of $1.3\cdot 10^4$).

[1] Random probe mean values are known to be opportunist. By definition they are minimizing the variance, and therefore do also decrease the measure of spread defined here.

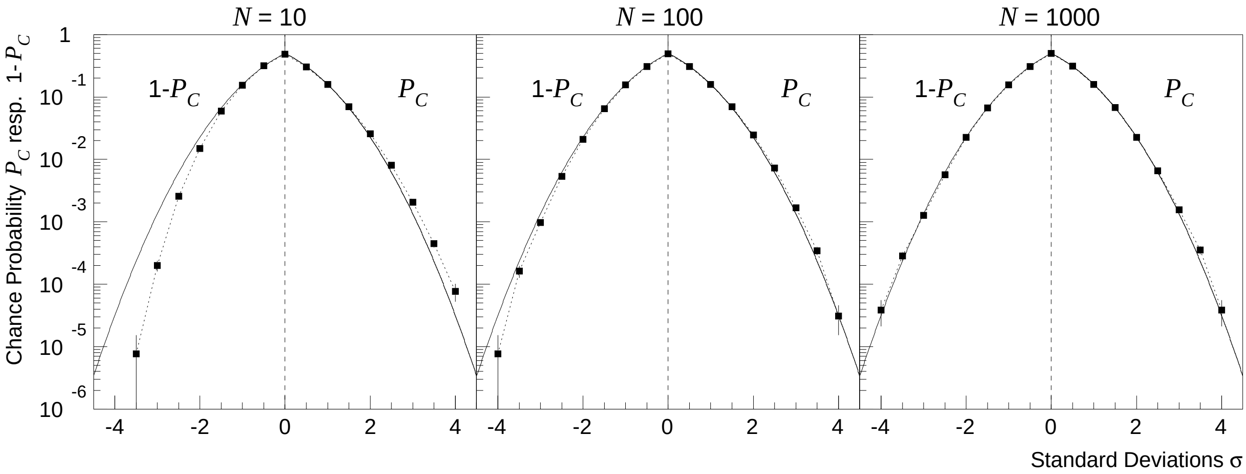

Fig. 2. Comparison of the M_N distribution with the normal distribution $N(1/e - \alpha/N, \beta/\sqrt{N})$: shown are the chance probabilities from the normal distribution (solid lines) and the relative frequencies from M$^{\mathrm{te}}$ Carlo calculations for M_N (symbols and dashed lines) against σ. Each sample distribution consists of $1.3 \cdot 10^5$ events.

of statistics, that for increasing N the distribution as a whole will tend towards a normal distribution. Therefore only two constants α, β need to be determined, for which the following asymptotic scaling laws apply:

$$\frac{1}{e} - \mu(\mathrm{M}_N) = \frac{\alpha}{N} \qquad \text{and} \qquad \sigma(\mathrm{M}_N) = \frac{\beta}{\sqrt{N}} \tag{11}$$

In simulations of Poisson processes by generating time sequences with the help of a pseudo random number generator, there could not be found any significant deviations from Eq. 11 for N ranging between 20 and 10^5 (using $1.3 \cdot 10^5$ individually generated sequences per considered number of N, see Fig. 1). With weighted means of the obtained values, α, β could be constrained to

$$\alpha = 0.189 \pm 0.004 \qquad \beta = 0.2427 \pm 0.0002 \tag{12}$$

There is only one faint (but significant) deviation from the $\sqrt{N}$ scaling law of the spread at $N = 10$ (the actually obtained number there is $\sigma \cdot \sqrt{10} = 0.2400 \pm 0.0005$). However, the use of the scaling law at $N = 10$ thus corresponds to an error of $\approx 1\%$ on a significance scale, which is negligible in almost all cases.

After having determined the mean values and the standard deviations with sufficiently high precision for most purposes, only the shape of the M_N distribution has to be compared to that of a normal distribution $N(1/e - \alpha/N, \beta/\sqrt{N})$ with the same mean and variance. In Fig. 2 the generated relative event frequencies are compared to $N(1/e - \alpha/N, \beta/\sqrt{N})$-distributed probabilities, for examples of $N = 10$, 100, 1000. Due to the fact that $\mu(\mathrm{M}_N) \approx 1/e$ is not in the middle of the allowed interval ($[0, 1[$) for M_N, a faint positive skewness of the M_N distribution can be noticed for smaller N. The peculiar drop of frequencies towards larger negative significances for $N = 10$ results from this, too, since $\mathrm{M}_{10} = 0$ corresponds to $\approx -4.5\,\sigma$. Such an effect is already hardly noticeable for $N = 20$. One notes that already for $N = 10$ and moderate positive deviations from the expectation value ($< 2\sigma$) the actual M_N distribution follows the normal distribution with a good precision, whereas for $N = 100$ the approximation with the normal distribution is globally valid for most practical cases. So only for extremely high precision requirements individual M$^{\mathrm{te}}$ Carlo calculations are necessary to assess the exact significance of an effect.

3. Using background events for the exp-test

The exp-test as developed so far can be readily applied in data analyses of experiments that have a sufficiently constant temporal acceptance. In the case of a varying acceptance function Eq. 2 is not valid, as one cannot expect a Poisson process any longer. (This case is sometimes referred to as *inhomogenous Poisson process*, see Chatfield 1994.)

If the temporal acceptance is discretely switched on and off, the problem can be easily solved by excluding the time intervals during which the data aquisition was actually switched off or blocked (e. g. due to electronics dead time). In the general case (continuously varying acceptance $a(t)$, which is the typical case in earth-bound γ-ray experiments), the variable t has to be replaced by a scaled effective time τ ($a(t) \cdot dt \longrightarrow d\tau$), in order to apply the exp-test as defined above, but this would require a precise and complete knowledge of the temporal acceptance function. However, a simple reasoning shows that background event times ($t_{\mathrm{BG},j}$) (being from a Poisson process in τ as well), simultaneously collected with the event times (t_i) of interest, could serve the same purpose if the background has a sufficiently high temporal density (e. g., in the case of cosmic rays, if the background events stem from a solid angle which is large compared to the one that defines the "on-source" events). In this case no explicit knowledge about $a(t)$ is required any longer. Instead, one can use the distribution of the ΔN_{BG} between two consecutive on-source events directly, thus testing, so to speak, whether the two distributions are mutually Poisson distributed.

Proceeding for this purpose from the pdf of the generalized time intervals $\Delta\tau$ (see Eq. 2) between two on-source events (the actual acceptance function $a(t)$ does not play a role in the end; the only exploited feature is that it scales the on-source and the background distributions simultaneously):

$$f_{\mathrm{on}}(\Delta\tau) = \frac{1}{C_{\mathrm{on}}} \exp\left(-\frac{\Delta\tau}{C_{\mathrm{on}}}\right) \tag{13}$$

with a certain, fixed C_{on}.
Now let the background events be from a Poisson process as well, with an expectation value λ_{BG} in a generalized time interval $\Delta\tau$ of

$$\lambda_{\mathrm{BG}}(\Delta\tau) = \Delta\tau / C_{\mathrm{BG}}$$

From Eq. 13 follows the distribution of the *mean values* for the number of background events between two on-source events:

$$f(\lambda_{\mathrm{BG}}) = \frac{1}{C} \cdot \exp\left(-\frac{\lambda_{\mathrm{BG}}}{C}\right) \tag{14}$$

(with a global expectation value of $C := C_{\mathrm{on}}/C_{\mathrm{BG}}$)
The complete pdf for the *number of background events between two consecutive on-source events* (in the following called "Inter-Events") can be derived now as a weighted (with $f(\lambda_{\mathrm{BG}})$) integration of the Poisson distribution $P_{\lambda_{\mathrm{BG}}}$ (see Eq. 1) over λ_{BG}:

$$\begin{aligned} w_C(n) &= \int_{\lambda_{\mathrm{BG}}=0}^{\infty} P_{\lambda_{\mathrm{BG}}}(n) \cdot f(\lambda_{\mathrm{BG}}) d\lambda_{\mathrm{BG}} \\ &= \frac{1}{C+1}\left(\frac{C}{C+1}\right)^n \end{aligned} \tag{15}$$

i. e. a discrete exponential distribution. Expressed with the help of the exponential function one yields

$$w_C(n) = \frac{1}{C+1} \cdot \exp\left(-\log\left(1+\frac{1}{C}\right) \cdot n\right) \tag{16}$$

(compare Eq. 2). Thus the number of background events can be used as a substitute for a generalized clock (measuring τ intervals). The precision of this "clock" will be defined by C (i. e. the ratio of the number of background events to the number of time intervals to test, or, equivalent, the mean value of all Inter-Events).

Analogously to the definitions in the last section let there be a frequency distribution $\mathcal{W}(n)$ of Inter-Events from a random probe of N intervals, with a mean value of

$$\sum_{n=0}^{\infty} n \cdot \mathcal{W}(n) = C^* \tag{17}$$

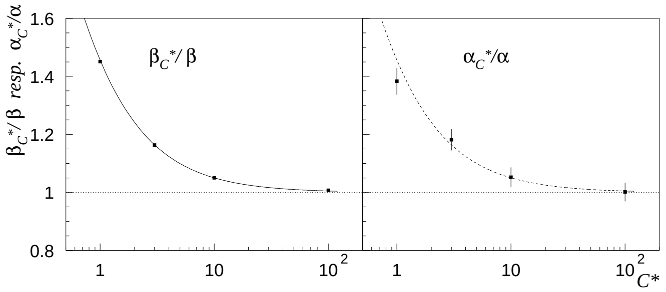

Fig. 3. The dependence of β_{C^*} and α_{C^*} from C^* from M$^{\mathrm{te}}$ Carlo calculations (see text). The solid line (left) shows the least squares fit (repeated with dashes at the right hand side). The sample sizes for each considered C^* are $1.8 \cdot 10^5$.

Eqs. 5, 6 are valid analogously, and one writes

$$\mathcal{H}(K) := \sum_{n=0}^{K} \left(1 - \frac{n}{C}\right) w_C(n) = \frac{K+1}{C+1}\left(\frac{C}{C+1}\right)^K \tag{18}$$
with $\mathcal{H}(-1) = \mathcal{H}(\infty) = 0$

with the maximum found at $K = [C]$.[2] The value at $[C]$ depends obviously on C itself, but tends again to $1/e$ if $C \to \infty$. With

$$\mathcal{M}(\mathcal{W}) := \sum_{n=0}^{[C^*]} \left(1 - \frac{n}{C^*}\right) \cdot \mathcal{W}(n) \qquad (C^* \text{ from Eq. 17}) \tag{19}$$

again $\mathcal{M}(\mathcal{W}) \in [0, 1[$, and for $\mathcal{W}(n)$ from Inter-Events in the case of a Poisson process, that $\mathcal{M}(\mathcal{W})$ has a mean value near

$$\mathcal{M}_0(C^*) = \frac{[C^*]+1}{C^*+1} \cdot \left(\frac{C^*}{C^*+1}\right)^{[C^*]} \tag{20}$$

and a variance that depends this time also on C^*.

Naming the mean value resp. the standard deviation of the corresponding $\mathcal{M}_{C^*,N}$ distribution $\mu_{C^*}(N)$ and $\sigma_{C^*}(N)$, respectively, and defining (analogously to Eq. 11)

$$\mathcal{M}_0(C^*) - \mu_{C^*}(N) = \frac{\alpha_{C^*}}{N} \qquad \sigma_{C^*}(N) = \frac{\beta_{C^*}}{\sqrt{N}} \tag{21}$$

then empirically the following parametrization can be found:

$$\alpha_{C^*} \approx \alpha \cdot k_1^{\frac{1}{C^*+k_2}} \qquad \beta_{C^*} \approx \beta \cdot k_1^{\frac{1}{C^*+k_2}} \tag{22}$$
with $k_1 = 1.67 \pm 0.02$ $\qquad k_2 = 0.37 \pm 0.03$

(α, β from Eq. 12)
The least squares fit that leads to k_1 and k_2 is shown in Fig. 3 (left). Since the spreads can by far more accurately be assessed than the mean values, again the fit to them provides the more precise numbers. However, in Fig. 3 (right) it can be seen that the dependency of

[2] Here and in the following the pair of square brackets [...] denotes the integer function.

α_{C^*}/α on C^* is compatible with the same parametrization, so that the same functional behaviour may well be assumed. Summarizing, it can be stated for the discrete case that was considered here, that the $\mathcal{M}_{C^*,N}$ distribution equals approximately a normal distribution $N(\mathcal{M}_0(C^*) - \alpha_{C^*}/N, \beta_{C^*}/\sqrt{N})$, with only small deviations similar to the ones shown in Fig. 2. For $C^* \to \infty$ the distribution of $\mathcal{M}_{C^*,N}$ tends in all aspects towards the M_N distribution. Since decreasing C^* lead to increasing spreads β_{C^*}, for optimization of the sensitivity C^* should be chosen as large as possible.

As a last general remark that applies to the continuous case as well as to the discrete case it should be stressed that the measure of the exp-test M(F) or $\mathcal{M}(\mathcal{W})$ can be assessed with running sums when one allows for two sequential passes over the data: in the first pass, C^* can be determined, whereas in the second pass the sum in Eq. 9 or Eq. 19 can be calculated. This possibility is a major advantage in comparison to binned procedures.

4. Sensitivity of the exp-test

This section deals with an application of the exp-test in a general scenario, in order to characterize its sensitivity.

Consider the case of a sporadically active source with a duty cycle $q \in]0,1]$, during which it "pollutes" the undisturbed background with additional events from another Poisson process (i. e., the constellation of the first question mentioned in the introduction). Let the total number of registered events be N, the number of background events be N_1, and the number of additional events from the source be N_2 (i. e. $N = N_1 + N_2$). Without loss of generality, the total observation time is set to 1 (see sketch in Fig. 4). The same pattern applies to the case of undiluted source events in a search for a signal of variablility (second question posed in the introduction), in which case N_1 denotes the quiescent flux Φ_{low} and $N_2/q + N_1$ corresponds to the flux Φ_{high} during the active state of an object.

To begin with the consideration of the continuous M_N statistics, it follows for the pdf $\tilde{F}$ of Δt to be

$$\tilde{F}(\Delta t) = \frac{N_1}{N}(1-q) \cdot f_{C_1}(\Delta t) + \frac{N_2 + qN_1}{N} \cdot f_{C_2}(\Delta t) \quad (23)$$

with $\quad C_1 = \dfrac{N}{N_1}$, $C_2 = \dfrac{qN}{N_2 + qN_1}$

and $\quad f_{C_i}(\Delta t)$ from Eq. 2

(i. e., according to Eq. 6, $C^* = 1$, without loss of generality)

From Eq. 9 it follows an expectation value for $\mathrm{M}(\tilde{F})$ of

$$\langle \mathrm{M}(\tilde{F}) \rangle = \frac{1}{e} \exp\left(\frac{N_2}{N}\right) \cdot \left(1 - q + q \exp\left(-\frac{N_2}{qN}\right)\right) \quad (24)$$

It is important to note that, due to the differential nature of the test, $\tilde{F}(\Delta t)$ and consequently $\langle \mathrm{M}(\tilde{F}) \rangle$ apply not only to a flux pattern as shown in Fig. 4, but

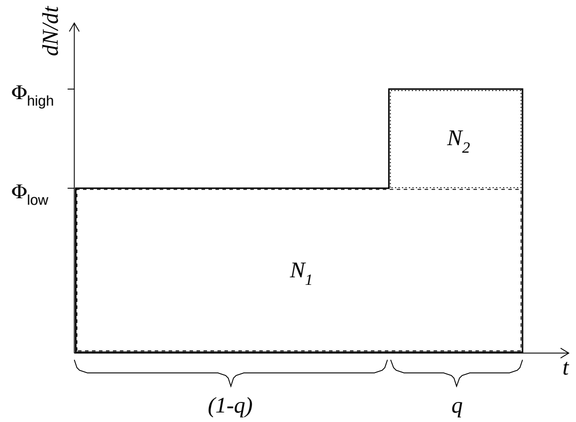

Fig. 4. Sketch of the mean event rates from a source with a variable flux (two flux levels).

to *all* two-leveled light curves. More specifically, they are neither affected by the actual position of the time window of the active state, nor by an arbitrary number of interrupts of the active state, as long as the total duty cycle is kept fixed and N_2 is large compared to the number of interrupts.

In the notion of the 1. question, one will ask for the *significance* of the exp-test result in comparison to the significance of a DC excess. For the exp-test, a significance S_{exp} of [3]

$$\langle S_{\text{exp}} \rangle \simeq \sqrt{N}(\langle \mathrm{M}(\tilde{F}) \rangle - 1/e)/\beta \quad (25)$$

can be expected. For any fixed N_2/N, $\langle S_{\text{exp}} \rangle$ is a positive, strictly monotone convex falling function of q, being zero at $q = 1$. In the limit of an arbitrary but fixed ratio $N_2/N \ll 1$ one gets from Eqs. 24, 25

$$\langle S_{\text{exp}} \rangle = \frac{1}{e\beta} \cdot \left\{1 - \frac{N}{N_2} q \left(1 - \exp\left(-\frac{N_2}{qN}\right)\right)\right\} \cdot \frac{N_2}{\sqrt{N}} \quad (26)$$

Under the above limit the expectation for the DC significance S_{DC} in the case of a perfect knowledge of the background level is

$$\langle S_{\text{DC}} \rangle \simeq N_2/\sqrt{N} \quad (27)$$

Thus

$$\langle S_{\text{exp}} \rangle = \underbrace{\frac{1}{e\beta}}_{\approx 1.51} \cdot \left\{1 - \frac{N}{N_2} q \left(1 - \exp\left(-\frac{N_2}{qN}\right)\right)\right\} \cdot \langle S_{\text{DC}} \rangle \quad (28)$$

i. e. in the limit $q \to 0$ (that means extremely burst-like behaviour of the source) one has to expect a significance

[3] In this consideration of expectation values there is no influence from the shift of the mean value α/N. In any case of real random probes with finite N, this shift cancels out since it concerns the actual probe as well as the expectation.

from the exp-test that is 50% higher than the DC result, so it can be stated that it is worth to apply the exp-test in the case when some DC excess is present from an appropriate candidate source. It should be emphasized once again that both significances are perfectly independent if the zero hypothesis is true.

Obeying the limit from above and defining $q_{\rm crit} := N_2/N$, for $q \ll q_{\rm crit}$ it follows from Eq. 28 that

$$\langle S_{\rm exp}\rangle = \frac{1}{e\beta}\cdot\left(1-\frac{q}{q_{\rm crit}}\right)\cdot\langle S_{\rm DC}\rangle \qquad (29)$$

while for $q = q_{\rm crit}$ one obtains

$$\langle S_{\rm exp}\rangle\Big|_{q=q_{\rm crit}} = \frac{1}{e}\langle S_{\rm exp}\rangle\Big|_{q=0} \qquad (30)$$

i. e. in the case of moderate DC significances, only for $q \lesssim q_{\rm crit}$ a significance from the exp-test that is worth mentioning can be expected. In the context of the first question, the exp-test therefore should be regarded as a test on burst-like temporal structures.

In the case of the discrete $\mathcal{M}_{C^*,N}$ statistics, qualitatively the same signature for the considered scenario occurs. One finds for the probability distribution

$$\widetilde{\mathcal{W}}(n) = \frac{N_1}{N}(1-q)\cdot w_{C_1}(n) + \frac{N_2+qN_1}{N}\cdot w_{C_2}(n) \qquad (31)$$

with $\quad C_1 = C^*\cdot\dfrac{N}{N_1}\ ,\ C_2 = C^*\cdot\dfrac{qN}{N_2+qN_1}$

and $\quad w_{C_i}(n)$ from Eq. 15

For arbitrary C^* it is not possible to give a closed expression for $\langle\mathcal{M}(\widetilde{\mathcal{W}})\rangle$ analogously to Eq. 24. However, the main feature of the discrete case can be assessed as follows. Using this time

$$\langle S_{\rm exp}\rangle \simeq \sqrt{N}(\langle\mathcal{M}(\widetilde{\mathcal{W}})\rangle - \mathcal{M}_0(C^*))/\beta_{C^*} \qquad (32)$$

one finds after lengthy but trivial calculations for integral C^* and $q \ll N_2/N \ll 1$:

$$\langle S_{\rm exp}\rangle \approx \left(\frac{C^*}{C^*+1}\right)^{C^*+1}\cdot\frac{1}{\beta_{C^*}}\cdot\left(1-\frac{q}{q_{\rm crit}}\right)\cdot\langle S_{\rm DC}\rangle \qquad (33)$$

Under the limitations stated above this yields for $C^* = 1$

$$\langle S_{\rm exp}\rangle \approx \underbrace{\frac{1}{4\beta_1}}_{\approx 0.7}\cdot\left(1-\frac{q}{q_{\rm crit}}\right)\cdot\langle S_{\rm DC}\rangle \qquad (34)$$

but for $C^* = 10$ already

$$\langle S_{\rm exp}\rangle \approx \underbrace{\left(\frac{10}{11}\right)^{11}\cdot\frac{1}{\beta_{10}}}_{\approx 1.4}\cdot\left(1-\frac{q}{q_{\rm crit}}\right)\cdot\langle S_{\rm DC}\rangle \qquad (35)$$

Please note in this context, that for the calculation of the DC significances a perfect knowledge of the background level was still assumed.

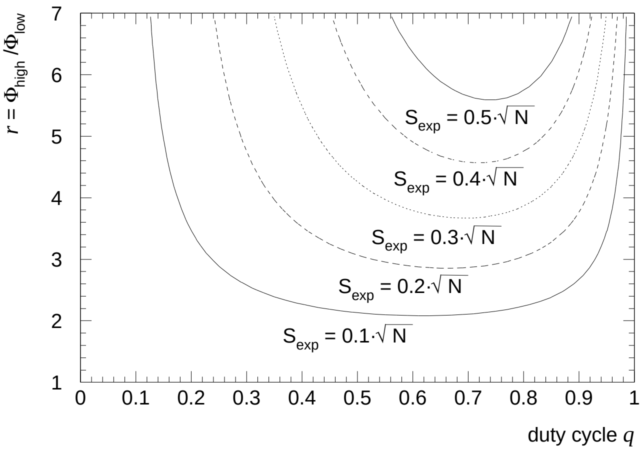

Fig. 5. Contour lines of the exp-test significance $S_{\rm exp}$ in the q-r plane. This figure allows for the determination of the detectable q-r region if the total number of events N and a required minimum value of $S_{\rm exp}$ are given.

The overall shape of the function $\langle S_{\rm exp}\rangle/\langle S_{\rm DC}\rangle$ in dependence of q is, apart from the lower absolute values, very similar to Eq. 28. It therefore can be summarized that the use of background events qualitatively yields the same results as the continuous M_N statistics, while the quantitative loss in sensitivity in dependence of C^* can be obtained from a comparison of Eq. 29 and Eq. 33.

Turning now to the second question mentioned in the introduction (pure source events), one will be interested in the sensitivity to detect a certain flux ratio $\Phi_{\rm high}/\Phi_{\rm low}$ when a duty cycle q is given, or vice versa. This shall only be carried out here for the continuous M_N statistics.

Defining $r := \Phi_{\rm high}/\Phi_{\rm low}$, one gets from Eq. 24:

$$\begin{aligned}\langle S_{\rm exp}\rangle &= \frac{\sqrt{N}}{\beta}\cdot\left(\langle\mathrm{M}(\tilde{F})\rangle - \frac{1}{e}\right)\\ &= \frac{\sqrt{N}}{e\beta}\cdot\left\{\left(1-q+q\cdot\exp\left(-\frac{1-r}{q+(1-q)r}\right)\right)\right.\\ &\qquad\left.\cdot\exp\left(\frac{q-q\cdot r}{q+(1-q)r}\right)-1\right\} \qquad (36)\end{aligned}$$

In Fig. 5 the resulting expectations for the significances are displayed as contour lines in the q-r plane. Making use of the fact that the the assumed two-level model is most efficient to produce an effect on $\mathrm{M}(\tilde{F})$ with respect to an upper limit of the ratio of maximum to minimum flux levels present for a given source, for a sufficiently high significance $S_{\rm exp}$ it is immediately possible to deduce a *lower limit* of actual flux variations $\Phi_{\rm max}/\Phi_{\rm min}$ for a source. For the given example values of $S_{\rm exp}$, these ratios can be read off from Fig. 5 as the ordinate values of the cusps of the displayed curves. For any given value of $S_{\rm exp}$, $\Phi_{\rm max}/\Phi_{\rm min}$ has to be obtained with graphical or numerical methods, regarding Eq. 36 as an implicit function $r(q)$ with S_{exp} as a parameter.

It can be concluded that in this case, where the significance is expressed in terms of flux levels in the high and the low states rather than in terms of event numbers, the exp-test is a quite broad-banded tool in the search for variabilities with respect to the duty cycle q.

5. Exp-test versus Kolmogorov test: a comparison

In order to recognize the strengthes of the exp-test on the one hand and the Kolmogorov test on the other, the latter shall be briefly reviewed here.

The Kolmogorov test uses the maximum D of the absolute differences of the cdf $G(t)$ of the temporal distribution of the event sequence and the expected cdf $g(t)$ (i. e. an equal distribution in the case of uniform temporal acceptance):

$$D := \max |G(t) - g(t)| \tag{37}$$

Being N the total number of events, it can be shown that the chance probabilities P_{kolmo} in dependence from D under the zero hypothesis are asymptotically distributed according to Kolmogorovs λ distribution:

$$P\left(D > \frac{\lambda}{\sqrt{N}}\right) \simeq 2 \cdot \sum_{k=1}^{\infty} (-1)^{k+1}\, e^{-2k^2\,\lambda^2} \tag{38}$$

(Kolmogorov 1933).
Consider now a distribution on $[0, 1]$ similar to the one shown in Fig. 4, with one uninterrupted activity phase of the length q, containing N_2 excess events, but this time with a specified starting time $t_0 \in [0, 1-q]$. The corresponding cdf is sketched in Fig. 6, and the expected difference $\langle\tilde{D}\rangle$ to the equal distribution can easily be calculated:

$$\langle\tilde{D}_{q,t_0}\rangle = (1 - q - \min(t_0, 1 - q - t_0)) \cdot \frac{N_2}{N} \tag{39}$$

Averaging over t_0 yields

$$\langle\tilde{D}_q\rangle = \frac{3}{4}\,(1-q) \cdot \frac{N_2}{N} \tag{40}$$

Again the question of the *significance*[4] one may expect from this constellation shall be posed. An easy estimate can be obtained under the assumption that $\tilde{D}$ already determines the actual D to a sufficient precision in the test, that means, a difference $D \geq \tilde{D}$ is fairly improbable under the zero hypothesis. Under this condition, in the series in Eq. 38 all but the first term are negligible, and it follows

$$P_{\text{kolmo}}\left(D > \tilde{D}_q\right) \approx 2 \cdot e^{-\frac{9}{8}(1-q)^2\,\frac{N_2^2}{N}} \tag{41}$$

[4] Calculating significances for results of the Kolmogorov test is rather unusual, but shall be performed here for the sake of easier comparison. With $\text{freq}(x) := \left(\sqrt{2\pi}\right)^{-1} \int_{-\infty}^{x} \exp(-z^2/2)\,dz$ every chance probability P_C corresponds to a significance of $S = -\text{freq}^{-1}(P_C)$.

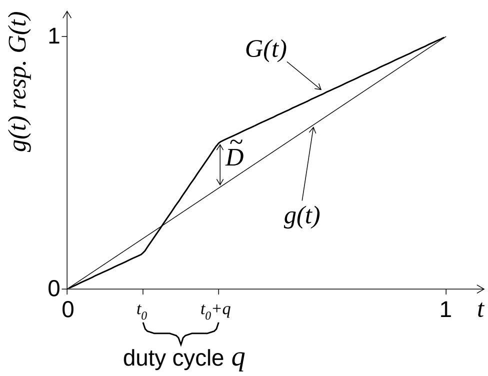

Fig. 6. Sketch of the cdf $G(t)$ of event times from a source being in an active state during $[t_0, t_0 + q]$, and the cdf $g(t)$ of the zero hypothesis (see text).

From

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{x} \cdot e^{-\frac{1}{2}x^2} &= \frac{1}{\sqrt{2\pi}} \cdot \int_x^{\infty} \left(1 + \frac{1}{z^2}\right) \cdot e^{-\frac{1}{2}z^2}\,dz \\ &= \left(1 + \mathcal{O}\left(\frac{1}{x^2}\right)\right) \cdot \underbrace{\frac{1}{\sqrt{2\pi}} \cdot \int_x^{\infty} e^{-\frac{1}{2}z^2}\,dz}_{P_C(x)}\end{aligned} \tag{42}$$

a first order asymptotic expansion of the standard normal distribution can be deduced:

$$P(S) \simeq \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{S} \cdot e^{-\frac{1}{2}S^2} \tag{43}$$

from which one obtains in a comparison with Eq. 41 for the significance S_{kolmo} of the Kolmogorov test result

$$\langle S_{\text{kolmo}}\rangle \approx \frac{3}{2}(1-q)\frac{N_2}{\sqrt{N}} \approx \frac{3}{2}(1-q)\langle S_{\text{dc}}\rangle \tag{44}$$

Numerical calculations show that this estimation is indeed approached for large N and $\langle S_{\text{dc}}\rangle$, while for small N and $\langle S_{\text{dc}}\rangle$ $\langle S_{\text{kolmo}}\rangle$ is somewhat smaller (Prahl 1999). For an example of $N = 100$ and $\langle S_{\text{DC}}\rangle = 3\,\sigma$ the significance $\langle S_{\text{kolmo}}\rangle$ is reduced by $\approx$ 20% for $q \to 0$. Nonetheless, Eq. 44 may serve as a rough estimate for all practical purposes.

It is obvious that for splitted activity intervals $\langle S_{\text{kolmo}}\rangle$ will be much lower. In a comparison with the exp-test (see Eq. 28) therefore it has to be stated that for *one single outburst* of a source with $q \ll N_2/N$ the expected significances from the Kolmogorov test and from the exp-test are similar (but slightly higher from the latter), while for a comparably long and uninterrupted activity interval the Kolmogorov test is more sensitive, whereas the exp-test is better suited to find some or many short outbursts.

Finally, the correlation of both tests under the zero hypothesis has to be studied. This is again performed in M$^{\text{te}}$ Carlo calculations. It turns out that the correlation

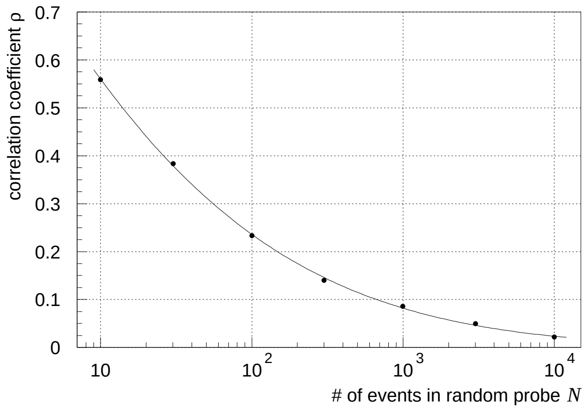

Fig. 7. The correlation coefficient between the significance result of the exp-test and that of the Kolmogorov test for M$^{\rm te}$ Carlo generated Poisson processes, versus the number N of tested events. The sample size for each considered value of N is $1 \cdot 10^5$.

coefficient ϱ between $S_{\rm kolmo}$ and $S_{\rm exp}$ is a monotonically decreasing function of N, being $\varrho \approx 0.56$ for $N = 10$, but already only $\varrho \approx 0.23$ for $N = 100$. The graph obtained for $\varrho(N)$ is shown in Fig. 7. One concludes that it is worth to apply both tests in a search of variable sources, maybe except for *very small* event numbers.

It should be mentioned that, like for the exp-test, the result of the Kolmogorov test is completely independent from the DC test result if the zero hypothesis is valid. Furthermore, the application of Smirnovs variant (Smirnov 1939) allows for the use of simultaneously acquired background events to define the zero hypothesis in the case of nonuniform temporal acceptance. Just like for the exp-test, the Kolmogorov test result can also be determined from runnig sums during two sequential passes over the experimental data.

6. Summary

It has been shown that, given N time intervals $\{\Delta T_k\}_{k=1\ldots N}$ between each pair of consecutive event times of the sequence to test (resp. N Inter-Events $\{n_k\}_{k=1\ldots N}$ in the discrete case), the quantity

$$\mathrm{M} = \frac{1}{N} \sum_{T_k < C^*} \left(1 - \frac{T_k}{C^*}\right)$$

$$\left(\text{resp.} \quad \mathcal{M} = \frac{1}{N} \sum_{n_k < C^*} \left(1 - \frac{n_k}{C^*}\right) \right)$$

is asymptotically normal distributed according to $N(1/e - \alpha/N, \beta/\sqrt{N})$ (resp. according to $N(\mathcal{M}_0(C^*) - \alpha_{C^*}/N, \beta_{C^*}/\sqrt{N}))$ if the temporal sequence represents a Poisson process (resp. inhomogenous Poisson process). The distribution of the time intervals $\mathrm{F}(\Delta t)$ resp. of the Inter-Events $\mathcal{W}(n)$ is easily accessible, and M(F) resp. $\mathcal{M}(\mathcal{W})$ are quickly calculable from it. In contrast to binned tests, the measure of the exp-test can be obtained from running sums when processing the data. The determination of the significance of the exp-test $S_{\rm exp}$ therefore is almost as easy as the calculation of a DC significance.

In observations that are limited by background fluctuations, obtained positive significances $S_{\rm exp}$ for a tested temporal sequence correspond to small chance probabilities and can directly be interpreted as significances for burst-like behaviour. For sufficiently small duty cycles, the significances from the exp-test are expected to be 1.5 times higher than the DC significances. The direct use of background events acquired simultaneously matches perfectly the requirements of typical experiments in the earthbound γ-ray astronomy.

Asking for signals of variability in pure source events, the result of the exp-test represents a simple and broadbanded measure for deviations from a constant flux. A positive result of the exp-test in such situations supplies an immediate information about the minimum flux variations $\Phi_{\rm max}/\Phi_{\rm min}$ present in the sample.

The results from the exp-test are by construction independent from a DC excess or deficit, and unaffected by interrupts of the activity cycle as far as possible. The test can be applied as soon as the number N of events to test surmounts ≈ 10.

Although every test on variability contains arbitrary elements, special care was taken to find a nonartificial measure that is suggested by itself when studying the elementary statistics of Poisson processes. From the viewpoint of the author, the exp-test fulfills this criterion.

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